Differential forms and κ-Minkowski spacetime from extended twist

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We unify κ-Minkowski spacetime, κ-Poincaré algebra and differential forms. We embed them into κ-deformed super-Heisenberg algebra related to bicrossproduct basis. Using twist, extended with Grassmann type variables, we obtain extended realization for κ-deformed coordinates, Lorentz generators and exterior derivative compatible with the symmetry algebra. Our results are relevant for constructing physical theories on noncommutative spacetime.

Keywords: noncommutative space, κ-Minkowski spacetime, differential forms, , super-Heisenberg algebra, realizations.

I. INTRODUCTION

The structure of spacetime at very high energies (Planck scale lengths) is still unknown and it is believed that, at these energies, gravity effects become significant and we need to abandon the notion of smooth and continuous spacetime. Among many attempts to find a suitable model for unifying quantum field theory and gravity, one of the ideas that emerged is that of noncommutative spaces [1]-[5]. Authors inclined to this idea have followed different approaches and considered different types of noncommutative (NC) spaces . In formulating field theories on NC spaces, differential calculus plays an essential role. The requirement that this differential calculus is bicovariant and also covariant under the expected group of symmetries leads to some problems.

One widely researched type of NC space, which is also the object of consideration in this letter, is the κ-Minkowski space [6]-[47]. This space is a Lie algebra type of deformation of the Minkowski spacetime

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and here the deformation parameter κ is usually interpreted as the Planck mass or the quantum gravity scale. κ-Minkowski space is also related to doubly special relativity [27]-[30]. For each NC space there is a corresponding symmetry algebra. In the case of κ-Minkowski space, the symmetry algebra is a deformation of the Poincaré algebra, known as the κ-Poincaré algebra. The κ-Poincaré algebra is also an example of a Hopf algebra. Some of the results of pursuing this line of research are, e.g., the construction of quantum field theories [31]-[38], electrodynamics [39]-[41], considerations of quantum gravity effects [42]-[44] and the modification of particle statistics [45]-[47] on κ-Minkowski space.

Regarding the problem of differential calculus on κ-Minkowski space, Sitarz has shown ([10]) that, in order to obtain bicovariant differential calculus, which is also Lorentz covariant, one has to introduce an extra cotangent direction. While Sitarz considered 3+1 dimensional space (and developed five dimensional differential calculus), Gonera et al. generalized this work to n dimensions in Ref. [11]. Another attempt to deal with this issue was made in [19] by the Abelian twist deformation of $U[\mathfrak{igl}(4,\mathbb{R})]$. Bu et al. in [20] extended the Poincaré algebra with the dilatation operator and constructed a four dimensional differential algebra on the κ-Minkowski space using a Jordanian twist of the Weyl algebra. Differential algebras of classical dimensions were also constructed in [17] and [18], from the action of a deformed exterior derivative.

In [25] the authors have constructed two families of differential algebras of classical dimensions on the κ-Minkowski space, using realizations of the generators as formal power series in a Weyl super-algebra. In this approach, the realization of the Lorentz algebra is also modified, with the addition of Grassmann-type variables. As a consequence, generators of the Lorentz algebra act covariantly on one-forms, without the need to introduce an extra cotangent direction. The action is also covariant if restricted to the κ-Minkowski space. However, one loses Lorentz covariance when considering forms of order higher then one.

Our motivation in this letter is to unify κ-Minkowski space-time, κ-Poincare algebra and differential forms. We embed them into κ-deformed super-Heisenberg algebra related to bicrossproduct basis. Using extended twist, we construct a smooth mapping between κ-deformed super-Heisenberg algebra and super-Heisenberg algebra. We present extended realization for κ-deformed coordinates, Lorentz generators and exterior derivative compatible with Lorentz covariance condition.

In section II, super-Heisenberg algebra is described. In section III, realization of κ-Minkowski space and κ-Poincare algebra related to bicrossproduct basis is given. In section IV, bicovariant differential calculus is analyzed. It is pointed out that there does not exist realization of Lorentz algebra in terms of commutative coordinates and momenta. In section V, κ-Minkowski space and NC forms are constructed by twist related to bicrossproduct basis. It is shown that consistency condition is not satisfied. In section VI, we present our main construction of κ-deformed super-Heisenberg algebra using extended twist. Extended realizations for Lorentz generators and exterior derivative invariant under symmetry algebra are presented. Finally, a short

conclusion is given.

II. SUPER-HEISENBERG ALGEBRA

In the undeformed case we consider space-time coordinates x_μ, derivatives $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, one forms $\mathrm{d}x_\mu \equiv \xi_\mu$ and Grassmann derivatives $q_\mu \equiv \frac{\partial}{\partial \xi^\mu}$ satisfying the following (anti)commutation relations:

$$\begin{aligned}&[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0, \quad [\partial_\mu, x_\nu] = \eta_{\mu\nu},\\ &\{\xi_\mu, \xi_\nu\} = \{q_\mu, q_\nu\} = 0, \quad \{\xi_\mu, q_\nu\} = \eta_{\mu\nu},\\ &[x_\mu, \xi_\nu] = [x_\mu, q_\nu] = [\partial_\mu, \xi_\nu] = [\partial_\mu, q_\nu] = 0,\end{aligned} \tag{1}$$

where $\mu = \{0, 1, 2, 3\}$ and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The algebra in (1) generates the undeformed super-Heisenberg algebra $\mathcal{SH}(x, \partial, \xi, q)$. The exterior derivative is defined as $\mathrm{d} = \xi_\alpha \partial^\alpha$, so $\xi_\mu = [\mathrm{d}, x_\mu]$.

We define the action $\triangleright : \mathcal{SH}(x, \partial, \xi, q) \mapsto \mathcal{SA}(x, \xi)$, where $\mathcal{SA}(x, \xi) \subset \mathcal{SH}(x, \partial, \xi, q)$. Super-Heisenberg algebra $\mathcal{SH}(x, \partial, \xi, q)$ can be written as $\mathcal{SH} = \mathcal{SA}\,\mathcal{ST}$, where $\mathcal{ST}(\partial, q) \subset \mathcal{SH}(x, \partial, \xi, q)$. For any element $f(x, \xi) \in \mathcal{SA}(x, \xi)$ we have

$$\begin{aligned}&x_\mu \triangleright f(x, \xi) = x f(x, \xi), \quad \xi_\mu \triangleright f(x, \xi) = \xi_\mu f(x, \xi),\\ &\partial_\mu \triangleright f(x, \xi) = \frac{\partial f}{\partial x^\mu}, \quad q_\mu \triangleright f(x, \xi) = \frac{\partial f}{\partial \xi^\mu}.\end{aligned} \tag{2}$$

The coalgebra structure of $\mathcal{ST}(\partial, q)$ is defined by undeformed coproducts:

$$\begin{aligned}&\Delta_0 \partial_\mu = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu, \quad \Delta_0 q_\mu = q_\mu \otimes 1 + (-)^{\mathrm{deg}} \otimes q_\mu,\\ &\mathrm{deg} = \xi_\alpha q^\alpha (\mathrm{mod} 2).\end{aligned} \tag{3}$$

The coalgebra structure with antipode and counit is (undeformed) super-Hopf algebra. Let us mention that super-Heisenberg algebra $\mathcal{SH}$ has also super-Hopf algebroid structure, which will be elaborated separately[4].

Now we introduce Lorentz generators $M_{\mu\nu}$:

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}, \tag{4}$$

with the following undeformed coproduct

$$\Delta_0 M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \tag{5}$$

and action $\triangleright$

$$\begin{aligned}&M_{\mu\nu} \triangleright x_\lambda = \eta_{\nu\lambda} x_\mu - \eta_{\mu\lambda} x_\nu, \quad M_{\mu\nu} \triangleright \xi_\lambda = \eta_{\nu\lambda} \xi_\mu - \eta_{\mu\lambda} \xi_\nu.\\ &M_{\mu\nu} \triangleright 1 = 0\end{aligned} \tag{6}$$

[4] for Hopf algebroid structure see [48], [49] and [26]

Using (5) and (6) we can derive the commutation relations

$$[M_{\mu\nu}, x_\lambda] = \eta_{\nu\lambda}x_\mu - \eta_{\mu\lambda}x_\nu, \quad [M_{\mu\nu}, \xi_\lambda] = \eta_{\nu\lambda}\xi_\mu - \eta_{\mu\lambda}\xi_\nu, \tag{7}$$

so that x_μ and ξ_μ transform as vectors (the same holds for ∂_μ and q_μ).

The Lorentz covariance condition

$$\begin{aligned} M_{\mu\nu} \triangleright f(x,\xi)g(x,\xi) &= m_0(\Delta_0 M_{\mu\nu} \triangleright f \otimes g) \\ M_{\mu\nu} \triangleright \mathrm{d}f(x,\xi) &= \mathrm{d}(M_{\mu\nu} \triangleright f(x,\xi)) \end{aligned} \tag{8}$$

(where m_0 is the multiplication map) implies

$$[M_{\mu\nu}, \mathrm{d}] = 0, \tag{9}$$

where $\mathrm{d}f(x,\xi) = \mathrm{d} \triangleright f(x,\xi) = [\mathrm{d}, f(x,\xi)] \triangleright 1$ and $\mathrm{d} \triangleright 1 = 0$. Note that the action $M_{\mu\nu} \triangleright f(x,\xi)$ in equation (8) is compatible with (6), (7) and

$$M_{\mu\nu} \triangleright f(x,\xi)g(x,\xi) = M_{\mu\nu} f(x,\xi)g(x,\xi) \triangleright 1 \tag{10}$$

The realization for $M_{\mu\nu}$ in $\mathcal{SH}(x,\partial,\xi,q)$ is

$$M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + \xi_\mu q_\nu - \xi_\nu q_\mu \tag{11}$$

and now it is easy to verify eqs.(4 - 7). Note that the Lorentz generators without the Grassmann-part $(\xi_\mu q_\nu - \xi_\nu q_\mu)$ can not satisfy the condition (9). Usually in differential geometry vector field $v = v_\mu\partial^\mu$ acts on a one-form $\xi_\beta = \mathrm{d}x_\beta$ as a Lie derivative $\mathcal{L}_v\xi_\beta = \mathrm{d}\mathcal{L}_v x_\beta = \mathrm{d}v_\beta$. In our approach the action through Lie derivative is equivalent to the action of $(v_\mu\partial^\mu + \mathrm{d}v_\mu q^\mu) \triangleright \mathrm{d}x_\beta = \mathrm{d}v_\beta$ and $[v_\mu\partial^\mu + \mathrm{d}v_\mu q^\mu, \mathrm{d}] = 0$.

III. κ-MINKOWSKI SPACE IN BICROSSPRODUCT BASIS

In κ-Minkowski space[5] with deformed coordinates $\{\hat{x}_\mu\}$ we have:

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = ia_0\hat{x}_i, \tag{12}$$

where a_0 is the deformation parameter. The deformed coproducts Δ for momentum generators p_μ and Lorentz generators $\hat{M}_{\mu\nu}$ in bicrossproduct basis [9] are

$$\begin{aligned} \Delta p_0 = p_0 \otimes 1 + 1 \otimes p_0, &\quad \Delta p_i = p_\mu \otimes 1 + e^{a_0 p_0} \otimes p_i, \\ \Delta\hat{M}_{i0} = \hat{M}_{i0} \otimes 1 + e^{a_0 p_0} \otimes \hat{M}_{i0} - a_0 p_j \otimes \hat{M}_{ij}, &\quad \Delta\hat{M}_{ij} = \hat{M}_{ij} \otimes 1 + 1 \otimes \hat{M}_{ij}. \end{aligned} \tag{13}$$

[5] Greek indices $(\mu, \nu, ...)$ are going from 0 to 3, and latin indices $(i, j, ...)$ from 1 to 3. Summation over repeated indices is assumed.

The algebra generated by $\hat{M}_{\mu\nu}$ and p_μ is called κ-Poincaré algebra where $\hat{M}_{\mu\nu}$ generate undeformed Lorentz algebra, p_μ satisfy $[p_\mu, p_\nu] = 0$ and the commutation relations $[\hat{M}_{\mu\nu}, p_\lambda]$ are given in [9]. Equations in (13) describe the coalgebra structure of the κ-Poincaré algebra and together with antipode and counit make the κ-Poincaré-Hopf algebra. We have the action[6] $\blacktriangleright: \hat{\mathcal{H}}(\hat{x}, p) \mapsto \hat{\mathcal{A}}(\hat{x})$, where $\hat{\mathcal{H}}(\hat{x}, p)$ is the algebra generated by $\hat{x}_\mu$ and p_μ and $\hat{\mathcal{A}}(\hat{x})$ is a subalgebra of $\hat{\mathcal{H}}(\hat{x}, p)$ generated by $\hat{x}_\mu$:

$$\begin{aligned} &\hat{x}_\mu \blacktriangleright \hat{g}(\hat{x}) = \hat{x}_\mu \hat{g}(\hat{x}), \quad p_\mu \blacktriangleright 1 = 0, \quad \hat{M}_{\mu\nu} \blacktriangleright 1 = 0 \\ &p_\mu \blacktriangleright \hat{x}_\nu = -i\eta_{\mu\nu}, \quad \hat{M}_{\mu\nu} \blacktriangleright \hat{x}_\lambda = \eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu. \end{aligned} \tag{14}$$

Namely, using coproducts (13) and action (14) one can extract the following commutation relations between $\hat{M}_{\mu\nu}$, p_μ and $\hat{x}_\mu$:

$$[p_0, \hat{x}_\mu] = -i\eta_{0\mu}, \quad [p_k, \hat{x}_\mu] = -i\eta_{k\mu} + ia_\mu p_k, \tag{15}$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\lambda] = \eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu - ia_\mu \hat{M}_{\nu\lambda} + ia_\nu \hat{M}_{\mu\lambda}. \tag{16}$$

The realization corresponding to bicrossproduct basis for $\hat{M}_{\mu\nu}$, p_μ and $\hat{x}_\mu$ in terms of undeformed x_μ and ∂_μ is[7] :

$$\begin{aligned} &\hat{x}_i^{(o)} = x_i, \quad \hat{x}_0^{(o)} = x_0 + ia_0 x_k \partial_k, \quad p_\mu = -i\partial_\mu \\ &\hat{M}_{i0}^{(o)} = x_i\left(\frac{1-Z}{ia_0} + \frac{ia_0}{2}\partial_k^2 - \frac{2}{ia_0}\mathrm{sh}^2(\frac{1}{2}A)Z\right) - (x_0 + ia_0 x_k \partial_k)\partial_i, \quad \hat{M}_{ij}^{(o)} = x_i\partial_j - x_j\partial_i, \end{aligned} \tag{17}$$

where $A = -ia_0\partial_0$ and $Z = e^A$ (for more details see [15] and [18]).

IV. BICOVARIANT DIFFERENTIAL CALCULUS

In the paper by Sitarz [10] there is a construction of a bicovariant differential calculus [50] on κ-Minkowski space compatible with Lorentz covariance condition (20), but with an extra one-form ϕ, which transforms as a singlet under the Lorentz generators. The algebra generated by $\hat{x}_\mu$ and one-forms $\hat{\xi}_\mu$, ϕ is closed in one-forms.

The deformed exterior derivative is defined by $[\hat{\mathrm{d}}, \hat{x}_\mu] = \hat{\xi}_\mu$, $\hat{\mathrm{d}}^2 = 0$ and satisfies ordinary Leibniz rule. Sitarz also assumes that the coproduct for Lorentz generator $\hat{M}_{\mu\nu}$ and momentum generator p_μ is in the

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[6] For more details see [23].

[7] where the superscript (o) denotes that the Lorentz generators and NC coordinates $\hat{x}$ are realized only in terms of undeformed x_μ and ∂_μ.

bicrossproduct basis (13), the action $\blacktriangleright$[8] is defined in (14) and it is extended to one-forms by:

$$\hat{\xi}_\mu \blacktriangleright 1 = \hat{\xi}_\mu, \quad \phi \blacktriangleright 1 = \phi$$
$$p_\mu \blacktriangleright \hat{\xi}_\nu = p_\mu \blacktriangleright \phi = \hat{M}_{\mu\nu} \blacktriangleright \phi = 0, \quad \hat{M}_{\mu\nu} \blacktriangleright \hat{\xi}_\lambda = \eta_{\nu\lambda}\hat{\xi}_\mu - \eta_{\mu\lambda}\hat{\xi}_\nu. \tag{18}$$

From coproducts (13) and eq.(18) we can find commutation relations $[\hat{M}_{\mu\nu}, \hat{\xi}_\lambda]$ and $[p_\mu, \hat{\xi}_\nu]$. In adition to eqs.(15) and (16) we have:

$$[p_\mu,\hat{\xi}_\nu] = [p_\mu, \phi] = [\hat{M}_{\mu\nu}, \phi] = 0,$$
$$[\hat{M}_{\mu\nu}, \hat{\xi}_\mu] = \eta_{\nu\lambda}\hat{\xi}_\mu - \eta_{\mu\lambda}\hat{\xi}_\nu. \tag{19}$$

Lorentz covariance condition

$$\hat{M}_{\mu\nu} \blacktriangleright \hat{f}(\hat{x},\hat{\xi})\hat{g}(\hat{x},\hat{\xi}) = m(\Delta\hat{M}_{\mu\nu} \blacktriangleright \hat{f}\otimes\hat{g})$$
$$\hat{M}_{\mu\nu} \blacktriangleright \hat{\mathrm{d}}\hat{f} = \hat{\mathrm{d}}(\hat{M}_{\mu\nu} \blacktriangleright \hat{f}) \tag{20}$$

implies

$$[\hat{M}_{\mu\nu}, \hat{\mathrm{d}}] = 0, \tag{21}$$

where $\hat{\mathrm{d}}\hat{f} = \hat{\mathrm{d}} \blacktriangleright \hat{f} = [\hat{\mathrm{d}}, \hat{f}] \blacktriangleright 1$ and $\hat{\mathrm{d}} \blacktriangleright 1 = 0$.

Sitarz claims that the algebra[9] between one-forms $\hat{\xi}_\mu$, ϕ and NC coordinate $\hat{x}_\mu$ that is compatible with (20 - 21) is given by

$$[\hat{x}_\mu, \phi] = \hat{\xi}_\mu, \quad [\hat{x}_0, \hat{\xi}_0] = -a_0^2\phi,$$
$$[\hat{x}_i, \hat{\xi}_j] = -ia_0\delta_{ij}(\hat{\xi}_0 + ia_0\phi), \quad [\hat{x}_0, \hat{\xi}_i] = 0, \quad [\hat{x}_i, \hat{\xi}_0] = -ia_0\hat{\xi}_i. \tag{22}$$

The realization for $\hat{x}_\mu$ is given in (17) and the realization for one-forms $\hat{\xi}_\mu$ and ϕ that satisfies (22) can be given in terms of undeformed x_μ, ∂_μ and ξ_μ[10]. The realizations[11] for one-forms and exterior derivative are

$$\hat{\xi}_0 = \xi_0\Big(1 + \frac{a_0^2}{2}\Box\Big) + ia_0\xi_k\partial_k, \quad \hat{\xi}_k = \xi_k - ia_0\xi_0\partial_k Z^{-1},$$
$$\phi = -\hat{\mathrm{d}}_s = \xi_0\Big(\frac{Z^{-1}-1}{ia_0} + \frac{ia_0}{2}\Box\Big) - \xi_k\partial_k, \quad \Box = \partial_i^2 Z^{-1} - \frac{4}{a_0^2}\mathrm{sh}^2(\frac{1}{2}A), \tag{23}$$

where we have denoted the exterior derivative for Sitarz's case with $\hat{\mathrm{d}}_s$.

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[8] Sitarz denotes this action with $\triangleright$

[9] The correspondence between algebra in [10] and (22) is $\frac{1}{\kappa} = -ia_0$, $x_\mu = \hat{x}_\mu$, $\mathrm{d}x_\mu = \hat{\xi}_\mu$, $\phi = \phi$, $N_i = \hat{M}_{0i}$, and $\hat{M}_i = \epsilon_{ijk}M_{jk}$.

[10] The algebra of undeformed operators is defined in Section II. and for $\blacktriangleright$ action we have $x_\mu \blacktriangleright 1 = \hat{x}_\mu$, $\xi_\mu \blacktriangleright 1 = \hat{\xi}_\mu$, $\partial_\mu \blacktriangleright 1 = 0$ and $q_\mu \blacktriangleright 1 = 0$.

[11] For more details see [25]

Relations (22) and (23) imply $\phi = -\hat{\mathrm{d}}_s$ and $\phi \triangleright 1 = 0$ so that the algebra[12] $\hat{\mathcal{SA}}$ is not isomorphic to $\mathcal{SA}^\star$. Also the problem with this construction is that the Lorentz generators $\hat{M}_{\mu\nu}$ can not be realized in terms of x_μ, ∂_μ, ξ_μ and q_μ in order to satisfy Lorentz covariance condition (20) which implies (21). If we take the realization (23) for $\hat{\mathrm{d}}_s$ and just want to find realization for $\hat{M}_{\mu\nu}$ so that (21) is fulfilled, then these $\hat{M}_{\mu\nu}$ don't satisfy Lorentz algebra (4).

Furthermore, in [25] differential algebras $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of classical dimension were constructed (avoiding the extra form ϕ), where all conditions were satisfied, except (16) and $\hat{M}_{\mu\nu}$ does not commute with exterior derivative. All these arguments lead to a conclusion that for the fixed realization (17) for $\hat{x}_\mu$, there is no realization for $\hat{M}_{\mu\nu}$ that satisfies κ-Poincaré-Hopf algebra (13) and Lorentz covariance condition (20), (21).

V. κ-MINKOWSKI SPACETIME FROM TWIST AND NC ONE-FORMS

In this section we will construct the noncommutative coordinates $\hat{x}_\mu$, coproducts and NC one-forms using twist operator.

A. κ-Minkowski spacetime from twist

We start with an Abelian twist (see [46], [19], [35] and [21])

$$\mathcal{F} = \exp(-A \otimes x_k \partial_k), \tag{24}$$

where $A = ia\partial = -ia_0\partial_0$. The bidifferential operator (24) satisfies all the properties of a twist (2-cocycle condition and normalization) and leads to noncommutative coordinates

$$\hat{x}_\mu = m_0\Big(\mathcal{F}^{-1} \triangleright (x_\mu \otimes id)\Big). \tag{25}$$

It follows that for this twist we get a realization for $\hat{x}_\mu$ exactly as in (17). The twist given by Eq. (24) also leads to an associative star product

$$f(x) \star g(x) = m_0\Big(\mathcal{F}^{-1} \triangleright (f \otimes g)\Big). \tag{26}$$

If we define the operators $M_{\mu\nu}$ as $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$, then $M_{\mu\nu}$ generate the undeformed Lorentz algebra, but their coproducts, obtained from the twist (24) do not close in the Poincaré algebra. For this reason we

[12] The algebra $\hat{\mathcal{SA}}$ is generated by $\hat{x}_\mu$, $\hat{\xi}_\mu$ and ϕ, and the algebra $\mathcal{SA}^\star$ is generated by x_μ and ξ_μ but with $\star$-multiplication. The star-product is defined by $f(x,\xi) \star g(x,\xi) = \hat{f}(\hat{x},\hat{\xi})\hat{g}(\hat{x},\hat{\xi}) \triangleright 1$.

consider the algebra $\mathfrak{igl}(4)$, generated by ∂_μ and $L_{\mu\nu} = x_\mu \partial_\nu$, which also has a Hopf algebra structure [24]. The coproducts of ∂_μ and $L_{\mu\nu}$ calculated as $\Delta\partial_\mu = \mathcal{F}\Delta_0 \partial_\mu \mathcal{F}^{-1}$, and analogously for $L_{\mu\nu}$, are

$$\Delta\partial_0 = \Delta_0\partial_0, \quad \Delta\partial_i = \partial_i \otimes 1 + e^A \otimes \partial_i \tag{27}$$

$$\Delta L_{ij} = \Delta_0 L_{ij}, \quad \Delta L_{00} = \Delta_0 L_{00} + A \otimes L_{kk} \tag{28}$$

$$\Delta L_{i0} = L_{i0} \otimes 1 + e^{-A} \otimes L_{i0}, \quad \Delta L_{0i} = L_{0i} \otimes 1 + e^A \otimes L_{0i} - ia_0 \partial_i \otimes L_{kk}. \tag{29}$$

Coproducts of the momenta p_μ, obtained from (27) by expressing p_μ in terms of ∂_μ ($p_\mu = -i\partial_\mu$), coincide with the coproducts of momenta in the bicrossproduct basis (13) (see also [9]). On the other hand, coproducts of the Lorentz generators $M_{\mu\nu}$, calculated from Eqs. (28) and (29) as $\Delta M_{\mu\nu} = \Delta L_{\mu\nu} - \Delta L_{\nu\mu}$, are different from the ones in the bicrossproduct basis (13) (more precisely, ΔM_{i0} is different), [24].

B. Noncommutative one-forms from twist

Our aim is to construct an exterior derivative $\hat{\mathrm{d}}$ and noncommutative one-forms $\hat{\xi}_\mu$ with the following properties

$$\hat{\mathrm{d}}^2 = 0, \quad [\hat{\mathrm{d}}, \hat{x}_\mu] = \hat{\xi}_\mu \tag{30}$$

$$\{\hat{\xi}_\mu, \hat{\xi}_\nu\} = 0, \quad [\hat{\xi}_\mu, \hat{x}_\nu] = K^\lambda_{\mu\nu}\hat{\xi}_\lambda, \ \ K^\lambda_{\mu\nu} \in \mathbb{C} \tag{31}$$

$$[\hat{\xi}_\mu, \hat{x}_\nu] - [\hat{\xi}_\nu, \hat{x}_\mu] = ia_\mu\hat{\xi}_\nu - ia_\nu\hat{\xi}_\mu \quad \text{(consistency condition)}, \tag{32}$$

where we have introduced $a_\mu = (a_0, \vec{0})$ so that (12) can be written in a unified way as

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu). \tag{33}$$

We want to find realization of NC one-forms in terms of undeformed algebra $\mathcal{SH}(x, \partial, \xi, q)$. If we now calculate $\hat{\xi}_\mu$, by analogy to Eq. (25), as $\hat{\xi}_\mu = m_0\big(\mathcal{F}^{-1} \triangleright (\xi_\mu \otimes id)\big)$, we get $\hat{\xi}_\mu = \xi_\mu$, so that the LHS of (32) equals to 0, while the RHS gives $ia_\mu\hat{\xi}_\nu - ia_\nu\hat{\xi}_\mu$ and the consistency condition is not fulfilled. Obviously we need to extend the twist defined in (24).

VI. EXTENDED TWIST

Our main goal is to construct a twist so that our bicovariant calculus satisfies the following properties:

1. The bicovariant calculus has classical dimension, i.e. there is no extra form like ϕ.

2. The algebra between $\hat{\xi}_\mu$ and $\hat{x}_\mu$ is closed in one-forms.

3. Generators $M_{\mu\nu}$ satisfy the Lorentz algebra.

4. The condition $[M_{\mu\nu}, \hat{\mathrm{d}}] = 0$, which is sufficient condition for (20), (21).

In order to satisfy all the requirements for $\hat{\xi}_\mu$ and $\hat{d}$ we define the extended twist

$$\mathcal{F}_{ext} = \exp\Big(-A \otimes (x_k\partial_k + \xi_k q_k)\Big). \tag{34}$$

This twist leads to

$$\hat{x}_i = m_0\Big(\mathcal{F}_{ext}^{-1} \triangleright (x_i \otimes id)\Big) = x_i, \quad \hat{x}_0 = m_0\Big(\mathcal{F}_{ext}^{-1} \triangleright (x_0 \otimes id)\Big) = x_0 + ia_0(x_k\partial_k + \xi_k q_k) \tag{35}$$

$$\hat{\xi}_\mu = m_0\Big(\mathcal{F}_{ext}^{-1} \triangleright (\xi_\mu \otimes id)\Big) = \xi_\mu. \tag{36}$$

Although the realization of $\hat{x}_0$ is changed with the addition of a term containing Grassmann variables, $\hat{x}_\mu$ still satisfy the same commutation relations Eq. (12), but the commutation relations between $\hat{x}_\mu$ and $\hat{\xi}_\mu$ are no longer all equal to 0

$$[\hat{\xi}_\mu, \hat{x}_i] = 0, \quad [\hat{\xi}_0, \hat{x}_0] = 0, \quad [\hat{\xi}_i, \hat{x}_0] = -ia_0\hat{\xi}_i. \tag{37}$$

Inserting (37) into (32) shows that in this case the consistency condition and the requirement from (31) are satisfied. Note that $\hat{x}_\mu$, $\hat{\xi}_\mu$, ∂_μ and q_μ generate the deformed super-Heisenberg algebra $\hat{\mathcal{SH}}$,which also has super-Hopf algebroid structure.

We now want to introduce an exterior derivative $\hat{\mathrm{d}}$ such that (30) is also fulfilled and gives rise to the same expression for $\hat{\xi}_\mu$ as (36). It is easily shown that this is achieved with

$$\hat{\mathrm{d}} = \xi^\alpha\partial_\alpha = \mathrm{d}. \tag{38}$$

Since $\hat{\xi}_\mu$ are undeformed, their $\triangleright$ and $\blacktriangleright$ actions are the same as for ξ_μ. Our construction can be extended to forms of higher order in a natural way. E.g., the space of two-forms can be defined as the space generated by $\hat{\xi}_\mu \wedge \hat{\xi}_\nu$. These two-forms then automatically satisfy

$$\hat{\xi}_\mu \wedge \hat{\xi}_\nu = \xi_\mu \wedge \xi_\nu = -\xi_\nu \wedge \xi_\mu = -\hat{\xi}_\nu \wedge \hat{\xi}_\mu. \tag{39}$$

Now we define the extended $\star$-product with

$$f(x,\xi) \star g(x,\xi) = m_0\Big(\mathcal{F}_{ext}^{-1} \triangleright f \otimes g\Big). \tag{40}$$

For $f(x)$ and $g(x)$, functions of x only, (40) coincides with (26), and if $f(\xi)$ and $g(\xi)$ are functions of ξ only, their extended $\star$-product is just the ordinary multiplication, $f(\xi) \star g(\xi) = f(\xi)g(\xi)$. As before (see (26)), the extended $\star$-product can be equivalently defined with the $\triangleright$ action: $f(x,\xi) \star g(x,\xi) = \hat{f}(\hat{x},\hat{\xi})\hat{g}(\hat{x},\hat{\xi}) \triangleright 1$.

In order to get a Lorentz i.e. $\mathfrak{igl}(4,\mathbb{R})$ covariant action, generators of $\mathfrak{gl}(4)$ also need to be extended. $L^{ext}_{\mu\nu}$ are defined by $L^{ext}_{\mu\nu} = x_\mu\partial_\nu + \xi_\mu q_\nu$. It can be easily checked that $L^{ext}_{\mu\nu}$, defined in this way, still satisfy the $\mathfrak{igl}(4)$ algebra, and furthermore, that they commute with d, $[L^{ext}_{\mu\nu}, \mathrm{d}] = 0$, so that

$$L^{ext}_{\mu\nu} \blacktriangleright \hat{\xi}_\lambda = [L^{ext}_{\mu\nu}, \hat{\xi}_\lambda] \blacktriangleright 1 = \mathrm{d}[L^{ext}_{\mu\nu}, \hat{x}_\lambda] \blacktriangleright 1 = \mathrm{d}(L^{ext}_{\mu\nu} \blacktriangleright \hat{x}_\lambda) = \hat{\xi}_\mu \eta_{\nu\lambda}. \tag{41}$$

The results for the coproducts, obtained from the extended twist, are

$$\Delta q_0 = \Delta_0 q_0, \quad \Delta q_i = q_i \otimes 1 + (-)^{deg} e^A \otimes q_i \tag{42}$$

$$\Delta L^{ext}_{ij} = \Delta_0 L^{ext}_{ij}, \quad \Delta L^{ext}_{00} = \Delta_0 L^{ext}_{00} + A \otimes L^{ext}_{kk} \tag{43}$$

$$\Delta L^{ext}_{i0} = L^{ext}_{i0} \otimes 1 + e^{-A} \otimes L^{ext}_{i0}, \quad \Delta L^{ext}_{0i} = L^{ext}_{0i} \otimes 1 + e^A \otimes L^{ext}_{0i} - ia_0\partial_i \otimes L^{ext}_{kk}. \tag{44}$$

The coproducts of ∂_0 and ∂_i, calculated in the same way, are again given by Eq. (27).

The action of $L^{ext}_{\mu\nu}$ on the product of $\hat{x}_\rho$ and $\hat{\xi}_\sigma$, calculated in three different ways:

(i) $L^{ext}_{\mu\nu} \blacktriangleright \hat{x}_\rho\hat{\xi}_\sigma = [L^{ext}_{\mu\nu}, \hat{x}_\rho] \blacktriangleright \hat{\xi}_\sigma + \hat{x}_\rho \blacktriangleright ([L^{ext}_{\mu\nu}, \hat{\xi}_\sigma] \blacktriangleright 1)$

(ii) $L^{ext}_{\mu\nu} \blacktriangleright \hat{x}_\rho\hat{\xi}_\sigma = (L^{ext}_{\mu\nu(1)} \blacktriangleright \hat{x}_\rho)(L^{ext}_{\mu\nu(2)} \blacktriangleright \hat{\xi}_\sigma)$

(iii) $L^{ext}_{\mu\nu} \blacktriangleright \hat{x}_\rho\hat{\xi}_\sigma = (L^{ext}_{\mu\nu(1)} \blacktriangleright \hat{x}_\rho)\mathrm{d}(L^{ext}_{\mu\nu(2)} \blacktriangleright \hat{x}_\sigma)$,

gives the same result, i.e., the action is in accordance with bicovariant calculus. One can easily show that neither the last equality, (iii), nor (41) would be satisfied had we used the ordinary definition of $L_{\mu\nu}$ ($L_{\mu\nu} = x_\mu\partial_\nu$). Similar expressions can be written in terms of the $\triangleright$ action and the extended $\star$ product.

The action (41) could also be obtained using the ordinary definition of $L_{\mu\nu}$ and promoting these generators to Lie derivatives. Using Cartan's identity, one would get

$$\mathcal{L}_{x_\mu\partial_\nu} \triangleright \xi_\lambda = d(\mathcal{L}_{x_\mu\partial_\nu} \triangleright x_\lambda) = d(x_\mu\eta_{\nu\lambda}) = \xi_\nu\eta_{\nu\lambda}. \tag{45}$$

However, there is a problem in this approach. Namely, promoting $L_{\mu\nu}$ to Lie derivatives would again give the realization (17) for $\hat{x}_\mu$ and in the case of deformed one-forms it would give $\hat{\xi}_\mu = \xi_\mu$, i.e., the consistency condition would not be fulfilled.

Finally, we present the realization of κ-Poincaré algebra compatible with bicovariant differential calculus in the bicrossproduct basis. Starting with the extended realization of $\hat{x}_\mu$ in (35) and demanding Lorentz algebra (4), (13) and (16) we find the realization for the Lorentz generators:

$$\begin{aligned} \hat{M}_{ij} &= x_i\partial_j - x_j\partial_i + \xi_i q_j - \xi_j q_i \\ \hat{M}_{i0} &= \hat{M}^{(0)}_{i0} + \xi_i(q_0 Z^2 + ia_0 q_k\partial_k) - [\xi_0 q_i + ia_0(\xi_k q_k \partial_i + \xi_k\partial_k q_i)] \end{aligned} \tag{46}$$

where $\hat{M}^{(0)}_{i0}$ is given in (17). The requirement that the Lorentz generators $\hat{M}_{\mu\nu}$ commutes with exterior derivatives $\tilde{\mathrm{d}}$, i.e. $[\tilde{\mathrm{d}}, \hat{M}_{\mu\nu}] = 0$ is fullfield for

$$\tilde{\mathrm{d}} = \frac{1}{1 + \frac{a_0^2}{2}\Box}\left[\xi_0\left(\frac{\mathrm{sh}A}{ia_0} + \frac{ia_0}{2}\partial_i^2 Z^{-1}\right) + \xi_j\partial_j Z\right] \tag{47}$$

All the details of this construction will be presented elsewhere.

In this letter we have shown that if the realization of κ-Minkowski algebra (12) is only in terms of Heisenberg algebra $\mathcal{H}(x, \partial)$, then there is no realization of Lorentz generators in order to satisfy all the requirements for bicovariant calculus ala Sitarz [10]. We have explicitly shown that if one wants to unify κ-Minkowski space-time, κ-Poincaré algebra and differential forms it is crucial to embed them into κ-deformed super-Heisenberg algebra $\hat{\mathcal{SH}}(\hat{x}, \hat{\xi}, \partial, q)$. This is done by using the extended twist. We choose bicrossproduct basis just as one example (e.g. to be consistent with Sitarz [10]), but similar constructions for other basis are also possible. Our main motivation for studying these problems is related to the fact that general theory of relativity together with uncertainty principle leads to NC space-time. In this setting, the notion of smooth space-time geometry and its symmetries are generalized using Hopf algebraic approach. Further development of the approach presented in this letter will lead to possible application to the construction of NCQFT (especially electrodynamics and gauge theories), quantum gravity models, particle statistics and modified dispersion relation.

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